Lecture 16 (Eulerian, Hamiltonian, and Planer graphs)

1 Eulerian Graphs

Definition 1. Let G be a graph. A closed trail $(v_0, v_1, \ldots, v_k, v_0)$ is called Eulerian trail if it contains all the edges of the graph. A graph G is said to be Eulerian if it has an Eulerian trail.

Theorem 1.1. Let G be a connected graph. Then the following statements are equivalent.

1. G is Eulerian.
2. Every vertex of G has even degree.
3. The set of edges can be partitioned into cycles.

Proof: $1 \Rightarrow 2$: Let G be Eulerian graph. Then G has an Eulerian trail T. Note that for each vertex v the trail enters through an edge and departs v from another edge. Thus at each stage, the process of coming in and going out contribute 2 to the degree of v. Since each edge appears exactly once, the degree of v is even.

$2 \Rightarrow 3$: Since G is connected and the degree of each vertex even, the graph is not a tree. So there is at least one cycle C_1 in G. If C_1 is not G. Let G_1 be the subgraph (possibly disconnected) of G after deleting the edges in C_1. Since each vertex in a cycle has degree 2, the degree of each vertex in G_1 has even and as before has a cycle C_2. Let $G_2 = G_1 - C_2$. We repeat the process of identifying the cycles until we get the graph $G_k = G - C_1 - C_2 - \cdots - C_k$ with no edges. Thus the set of edges of these cycles gives the required partition.

$3 \Rightarrow 1$: Suppose the set of edges in a connected graph G is the disjoint union of k cycles. Consider any one of these cycles, say cycle C_1. Since the graph is connected, there is a cycle, say C_2, such that the two cycles have a vertex v_1 in common. Let Q_{12} be the circuit that consists of all the edges in these two cycles. As before, there is a cycle C_3 such that this cycle and the circuit Q_{12} have no edge common but do have vertex v_2 in common. Let Q_{123} be the circuit that contains all the edges of these three edge-disjoint cycles. We repeat this process until we get a circuit that contains all the edges of the graph. Thus graph is Eulerian.

2 Hamiltonian Graph

Definition 2. Let G be a graph. A cycle in G is said to be Hamiltonian if it contains all vertices of G. If G has a Hamiltonian cycle, then G is called a Hamiltonian graph.

Example 1.

1. *For each positive integer $n \geq 3$, the cycle C_n is Hamiltonian.*
2. *For each positive integer $n \geq 3$, the cycle K_n is Hamiltonian.*
3. *The graphs corresponding to all platonic solids are Hamiltonian.*

Proposition 1. *The Petersen graph is not Hamiltonian.*

Proof: Suppose G is Hamiltonian. So, G contains cycle $C_{10} = (1, 2, 3, \ldots, 10, 1)$ as a subgraph. Note that degree of each vertex 3 and $g(G) = 5$. Consider, the vertices 1,2 and 3.

In the view of $g(G)$, the vertex 1 can be adjacent to only one of the vertices 5,6, or 7. If 1 is adjacent to 5, then the possible third vertex that is adjacent to 10 will crate cycles of length 3 or 4. Similarly, if 1 is adjacent to 7 then there is no choice for the possible third vertex that can be adjacent to 2. So, let 1 be adjacent to 6. Then, 2 must be adjacent to 8. In this case, note that there is no choice for the third vertex that can be adjacent to 3. Thus, the Petersen graph is non-Hamiltonian.

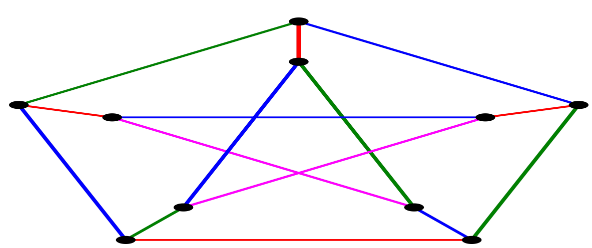

Figure 1: Petersen graph

Theorem 2.1. (Ore's Theorem: A sufficient condition for a graph to be Hamiltonian). A simple graph with n vertices (where $n > 2$) is Hamiltonian if the sum of the degrees of every pair of non-adjacent vertices is at least n.

Theorem 2.2. (Dirac's Theorem: A sufficient condition for a graph to be Hamiltonian). A simple graph with n vertices (where $n > 2$) is Hamiltonian if the degree of every vertex is at least $n/2$.

Proof: If each degree is at least $n/2$, the sum of every pair of vertices is at least n. Then the proof follows by Ore's theorem.

3 Planar Graph

A graph is said to be embedded on a surface S when it is drawn on S so that no two edges intersect, except at end points. A graph is said to be planar if it can be embedded on the plane.

Examples:

1. A tree is embeddable on a plane.
2. Any cycle C_n, $n \geq 3$ is planar.
3. The K_4 is planar.
4. The $K_{2,3}$ is planar.
5. Draw a planar embedding of $K_5 - e$, where e is any edge.
6. Draw a planar embedding of $K_{3,3} - e$, where e is any edge.

Definition 3. Consider a planar embedding of a graph G. The regions on the plane defined by this embedding are called faces/regions of G. The unbounded face/region is called the exterior face.

Theorem 3.1. Let G be a connected planar graph with V number of vertices, E number of edges and F number of faces. Then $V - E + F = 2$.

Proof: We use induction on E. Let $E = 1$. Then $V = 2$ and $F = 1$ and hence $V - E + F = 2$.

Suppose the result holds for all the planar-connected graphs with $E \leq n - 1$.

Let G be a connected planar graph with $E = n$. Then we have two cases:

Case 1: If G has no cycles. Then G is a tree and hence the result is true.

Case 2: Suppose G contains at least one cycle. Let e be an edge in a cycle and $G' = G - e$. Then $E' = E - 1$, $V' = V$ and $F' = F - 1$. Then $E' - V' + F' = 2$ implies $V - E + F = 2$. Thus the result holds.

Corollary 1. If G is a connected planar simple graph with E edges and V vertices, where $V \geq 3$, then $E \leq 3V - 6$.

Proof: Each face has at least 3 edges and each edge is participated in two faces. So, the number of edges $E \geq 3F/2$. Now the proof follows by $V - E + F = 2$.

Exercise: K_5 is non-planar, because here $V = 5$ and $E = 10$, using the above result leads to $E = 10 \leq 3 * 5 - 6 = 9$, a contradiction.

Corollary 2. If a connected planar simple graph has E edges and V vertices with $V \geq 3$ and no circuit of length three, then $E \leq 2V - 4$.

Proof: Note that $E \geq 2F$. Then the proof follows by $V - E + F = 2$.

Exercise: $K_{3,3}$ is non-planar, because $K_{3,3}$ has no circuit of length 3 with $V = 6, E = 9$, which using the above result leads to $E = 9 \leq 2 * 6 - 4 = 8$, a contradiction.

Definition 4. Let G be a graph. Then, a subdivision of an edge $u\,v$ in G is obtained by replacing the edge by two edges $u\,w$ and $w\,v$, where w is a new vertex. Two graphs are said to be homeomorphic if they can be obtained from the same graph by a sequence of subdivisions.

Theorem 3.2. (Kuratowski's Theorem) A graph is non planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

Exercise: Petersen graph is non-planar. It can be shown that the said graph is homeomorphic to $K_{3,3}/K_5$.